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No. 778

METHOD FOR THE DETERMINATION OF THE SPANWISE
LIFT DISTRIBUTION

By A. Lippisch

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SUMMARY

There are now several methods by means of which the lift distribution of a wing can be determined; but when attempting to apply these methods in practice, it is found that they involve, besides considerable loss of time, expert knowledge such as a practical engineer may not always be presumed to know. So in order to remedy this fault, a new method has been developed which even in first and second approximations yields results adequate for practical purposes.

As in the majority of the other methods, this method is likewise based on the Fourier series for the representation of the lift distribution. The lift distribution, as well as the angle of attack, is split up in four elementary distributions. The insertion of the angle-of-attack distribution in the Fourier series for the lift distribution gives a compound third series which is of particular advantage for the determination of the lift distribution. For, employing the Fourier series itself with a view of representing the lift distribution, involves the calculation of a greater number of coefficients necessary for compensating the ensuing oscillations. The function formulated from the Fourier series and the angle-of-attack distribution removes this fault. By including the angle-of-attack distribution, the effect of the higher terms of the Fourier series on the shape of the lift distribution is so small that only a few coefficients afford an adequate approach. The method is illustrated in an example and supplemented by a graphical method. Lastly, the results of several comparative calculations with other methods are reported.

*"Verfahren zur Bestimmung der Auftriebsverteilung längs Spannweite." Luftfahrtforschung, June 17, 1935, pp. 89-106.

I. INTRODUCTION

The chief aim in airplane development is greater flight performance, as evidenced by the efforts of the modern airplane designers to minimize the residual drag as much as possible. This brings the useful wing drag always more and more to the fore. Whereas, in the past, this wing drag was, on the whole estimated, it now is common practice to define the effect of every single wing element in order to make the determination of the best shape of the wing for the given design problem possible. Aside from the flight performances, there are yet a number of other factors vitally affecting the design of the wing shape, namely, the desired flying qualities on the one hand, through which the pitch, yaw, as well as the roll stability are influenced; while, on the other hand, the designer will strive to bring the fundamental principles for the aerodynamically best wing shape into accord with the static requirements. As the lift distribution simultaneously supplies the basis for the load distribution in the various operating conditions, both problems may be directly connected. Lastly, there are the conditions of manufacture and service, with a view to simplicity and the special service qualities for a particular purpose.

All these factors affect the design, and it is the problem of the aerodynamic calculation of the wing to define these effects of such measures and to reconcile the different antagonistic requirements.

This is being illustrated in several examples. The best lift distribution for finite span is, as is known, the elliptical. When other limiting conditions, such as optimum moment distribution, for example, are demanded, they may be found from Prandtl's derivations (reference 1). It further may be presumed, as known, that a minor divergence from the pure elliptical lift distribution, as occurs on the conventional wings, has no substantial effect on the induced drag. The effect of change of lift distribution on the induced drag has been pretty well explored.

But it is important to establish whether the used wing form in extreme positions would reveal vitiating flight characteristics as a result of the applied lift distribution. Such a study is essential on tapered wings, for instance. The tapered wing in figure 1, with constant profile and angle of attack, has dashed lift distribution.

Defining therefrom the specific loading of the separate wing elements, i.e., the lift coefficient c_a , it is found that the curve of c_a along the span reveals a distinct maximum toward the tips. In operating conditions with high total lift the breakdown of the flow then starts in the vicinity of this maximum. As even the least unsymmetry in the air flow or in the design would induce a one-sided breakdown of the flow at a wing, such a wing would be laterally and directionally unstable at stalling. Thus, the lift distribution must be modified so as to shift the maximum c_a toward the center of the wing. This is accomplished by twisting the wing tips, and figure 1 shows that the intended effect has been actually obtained. The result on the lift distribution is small and practically zero as far as the flight performances are concerned.

As previously pointed out, knowledge of the lift distribution under the various operating conditions is an absolutely necessary basis for the stress analysis. It was common practice heretofore to employ empirical load assumptions and to stipulate that they agree with the actual loads. With certain limitation, this empiricism may prove true for the straight, nontwisted wing, but for twisted wings it leads to erroneous conclusions which may end in too small dimensions of the structural parts of the wing. The lift distribution for various flight attitudes, expressed in lift coefficient of the total wing $c_{a_{total}}$, is shown in figure 2 for the wing of figure 1. In the various flight attitudes the steady dynamic pressure now changes in such a way that a drop in c_a is accompanied by a marked rise in dynamic pressure. Thus when plotting the load-distribution curves - the course of $c_a t q$ - it is readily seen that the load distributions undergo considerable changes in the different operating conditions. One striking fact is that the bending moments in diving and inverted flight induce enormous stresses at the tips, with the result that spar dimensions based on an empirical load assumption, would not be safe.

It would lead too far afield to touch upon the aileron effect and torque. Suffice it to say that the seemingly statically favorable wing tip modified according to aerodynamic viewpoints, would lose a large portion of these advantages.

In many cases the smooth contour of the wing is interrupted by cutaway sections or superstructures. Such

local changes in chord distribution must be reconciled with a stipulated lift distribution in order to avoid sudden lift interferences in the range of normal operating conditions. Muttray's (reference 2) explanation on this subject is that the local chord change must be compensated by an opposite angle of attack change, or change of camber. The middle piece of the wing with cutaway, in figure 3, must obtain higher angles of attack, or profiles with greater mean camber.

The inverse process is applicable to wing-body fillets or cowlings. The amount of this angle of attack and profile change can be implicitly deduced from the lift-distribution calculation, and the designer is in position to effectuate the desired wing changes.

The few instances cited, already attest to the necessity of lift-distribution calculations. When, in spite of this fact, the application of this knowledge derived from airfoil theory is not common practice with the productively engaged airplane builder, the main reason for this lack is attributable perhaps to the enormous paper work involved with the conventional methods and in most cases also to the lack of personnel adequately trained to do such work.

The practical airplane designer needs a method which on first approach affords serviceable results and which, without the use of complicated equations, conforms to actual practice.

II. GENERAL PRINCIPLES FROM AIRFOIL THEORY

Notation

Γ , circulation
 c_a , lift coefficient
 c_{m_q} , rolling-moment coefficient
 c_{m_s} , yawing-moment coefficient
 F , wing area (m^2)
 b , span (m)

- t , wing chord (m)
 y , abscissa along the span
 v , flow velocity at infinity (m/s) (flight speed)
 w , induced downwash at wing (m/s)
 α , angle of attack referred to original flow direction
 α_i , induced angle of attack at wing
 $2\pi\eta = \left[\frac{\partial c_a}{\partial \alpha} \right]_{\infty}$ = lift increase in 2-dimensional flow
 ρ , air density $\left[\frac{\text{kg}}{\text{m}^3} \frac{\text{s}^2}{\text{m}^2} \right]$
 φ , angle abscissa of span defined with:

$$\cos \varphi = - \frac{y}{b/2}$$

The lift of the individual wing element is a function of Γ , that is,

$$A = \rho v \Gamma$$

is the lift per unit length.

Accordingly,

$$\Gamma = c_a t \frac{v}{2}$$

because

$$A = c_a t \frac{\rho}{2} v^2$$

is the lift per unit length.

The circulation distribution is a function of the span

$$\Gamma = F(y)$$

The rate of downwash at point y_p due to Γ is:

$$w_p = \frac{1}{4\pi} \int_{-b/2}^{+b/2} \frac{\frac{\partial \Gamma}{\partial y}}{y_p - y} dy$$

Owing to the induced rate of downwash the angle of attack at the point of the wing decreases by α_i . It is:

$$\tan \alpha_i = \frac{w}{v}$$

where, since the angle is small, the tangent is replaced by the angle itself ($\tan \alpha_i \cong \alpha_i$).

The above term substituted for Γ gives

$$\alpha_i = \frac{1}{8\pi} \int_{-b/2}^{+b/2} \frac{\frac{\partial (c_a t)}{\partial y}}{y_p - y} dy \quad (1)$$

and the lift distribution in place of the circulation distribution gives

$$c_a t = f(y)$$

The lift coefficient c_a itself is a function of the effective angle of attack. It is

$$\left. \begin{aligned} c_a &= 2\pi\eta\alpha_{\text{effective}} \\ &= 2\pi\eta(\alpha - \alpha_i) \end{aligned} \right\} \quad (2)$$

whence the integral equation for the lift distribution

$$c_a t = 2\pi\eta t \left[\alpha - \frac{1}{8\pi} \int_{-b/2}^{+b/2} \frac{\frac{\partial (c_a t)}{\partial y}}{y_p - y} dy \right] \quad (3)$$

or

$$\begin{aligned} c_a t = 2\pi\eta t \left[\alpha - \frac{\eta}{4} \int_{-b/2}^{+b/2} \frac{\frac{\partial (\alpha t)}{\partial y}}{y_p - y} dy \right. \\ \left. + \frac{\eta}{4} \int_{-b/2}^{+b/2} \frac{\frac{\partial (\alpha_i t)}{\partial y}}{y_p - y} dy \right] \quad (3a)^* \end{aligned}$$

*See footnote, page 7.

III. ORDER OF LIFT AND ANGLE-OF-ATTACK DISTRIBUTION

The lift grading given through $c_a t$ may be visualized as the sum of elementary distributions, so that each elementary grading may be treated separately and any total grading may be obtained for the desired operating conditions. These elementary distributions are (see fig. 4):

1. The normal distribution $c_a t_{\text{normal}}$, that is, that lift distribution for which angle α is the same at any point of the wing; that is, $\alpha = \text{constant}$. This distribution is contingent upon the plan form of the wing and changes with α , and/or the total lift.

2. The zero distribution $c_a t_{\Delta}$, which is produced through an equilateral - partly positive, partly negative - angle-of-attack curve. The induced positive and negative partial lifts cancel, so that the total lift of the zero distribution of a semi-wing is always equal to zero. The angle-of-attack curve therefore corresponds to an equilateral twist. The aspect of the zero distribution is governed by plan form and twist, although unaffected by changes in total lift, and total angle of attack of the wing.

3. The normal roll distribution $c_a t_Q$: This quota of the lift distribution is due to a rolling motion and may be visualized as being due to a straight, twisting curve. The pertinent α is directly proportional to the distance of the particular wing element from the axis of roll, that is, $\frac{\alpha_Q}{y} = \text{constant}$. The curve of normal roll distribution is dependent on the plan form of the wing and varies as the rolling moment, or $\frac{\alpha_Q}{y}$.

4. The roll-zero distribution $c_a t_{\Delta_Q}$: This portion of the lift distribution arises from a steady roll of the wing about its longitudinal axis with corresponding aileron deflection. The rolling moment due to aileron deflection is compensated as a result of the induced rolling motion, so that the lift distribution produced by it sets up

*(From p. 6.) The treatment of the problem through further reduction of this integral equation is reserved for a future report.

no rolling moment. The roll-zero distribution is dependent on plan form and ailerons, and changes with the aileron setting.

These four elementary distributions constitute the total distribution of an operating condition. If the flight attitude is one without aileron deflection the roll distribution does not take place; the normal distribution and zero distribution in twist make it possible to determine any and all operating conditions within the range of straight lift increase. For flight attitudes in stalling range, the data from the straight lift increase are applicable but only with certain restrictions, although such flight attitudes lend themselves also to mathematical treatment, as shown elsewhere. If an aileron deflection is present it results, for the time, in an additional lift distribution which creates the particular aileron rolling moment. The amount of this rolling moment governs the magnitude of the normal roll distribution. If the roll-zero distribution becomes additive to the particular aileron deflection, it yields the additional aileron lift distribution at the inception of roll. During the subsequent roll a moment compensation takes place which renders the rolling moment zero in the final attitude; that is, the normal roll distribution likewise becomes zero. To this the roll-zero distribution, conformably to the aileron deflection is then simply added.

These four elementary distributions have four equivalent angle-of-attack proportions. They are:

1. The mean angle of attack α_m , equal at any point of the wing and directly proportional to the total lift. If $\alpha_m = 0$, the total lift = 0 also.

2. Angle of twist Δ , which corresponds to an equilateral wing twist producing, for instance, positive Δ in the center section and negative Δ at the wing tips. The curve of this angle being the result of constructional measures (modified profile or incidence), it cannot be changed during flight unless flaps fitted on the wing can be deflected equally (landing flaps, trimming flaps, differential ailerons).*

*The aileron setting with differential operation corresponds to the deflection without differential for equal upward pull of both ailerons.

3. The mean angle of roll α_Q , which, as previously stated, is directly proportional to the distance from mid center, or in other words, α_Q is always a straight line through the origin; and $\frac{\alpha_Q}{y}$ has the same value at any point of the wing.

4. Angle of twist in roll Δ_Q , produced as difference between the additional angle of attack due to aileron deflection and that mean angle of roll which would create equal rolling moment. Consequently, Δ_Q produces an additional lift distribution without rolling moment.

IV. DEVELOPMENT OF METHOD

1. General Derivation

The various lift distributions were computed with the well-known Fourier series. Since this distribution changes with the scale of the wing as well as the dynamic pressure, the semispan is chosen as unit scale and the dynamic pressure is put $q = 1$. The conversion of the obtained results to the actual dimensions and speeds is effected through expansion with $b/2$ and q . Thus,

$$q = 1; \quad \frac{b}{2} = 1$$

and all length dimensions are reduced to unit scale by dividing with $b/2$. The span coordinate y then, is from -1 (left wing tip) over zero (wing center) toward +1 (right wing tip). The integral for determining α_i is then taken within the -1 to +1 range.

Expanded in Fourier series, the lift distribution is:

$$\begin{aligned} c_a t &= A_1 \sin \varphi + A_2 \sin 2\varphi + A_3 \sin 3\varphi + \dots \\ &= \sum_{n=1}^{\infty} A_n \sin n \varphi \end{aligned} \quad (4)$$

wherein the angle of abscissa φ is tied to the span abscissa through

$$\cos \varphi = -y$$

$$\sin \varphi = \sqrt{1 - y^2}$$

The induced angle of attack is:

$$\alpha_i = \frac{1}{8 \sin \varphi} [A_1 \sin \varphi + 2A_2 \sin 2\varphi + 3A_3 \sin 3\varphi + \dots]$$

$$= \frac{1}{8 \sin \varphi} \sum_1^{\infty} n A_n \sin n \varphi \quad (5)$$

Then the angle of attack is defined as:

$$\alpha = A_1 \frac{\sin \varphi}{\sin \varphi} \left[\xi + \frac{1}{8} \right] + A_2 \frac{\sin 2\varphi}{\sin \varphi} \left[\xi + \frac{2}{8} \right] +$$

$$+ A_3 \frac{\sin 3\varphi}{\sin \varphi} \left[\xi + \frac{3}{8} \right] + \dots$$

$$= \sum_1^{\infty} A_n \frac{\sin n \varphi}{\sin \varphi} \left[\xi + \frac{n}{8} \right] \quad (6)$$

with $\xi = \frac{\sin \varphi}{2 \pi t \eta}$

that is, a variable dependent on the chord distribution and the lift efficiency of the individual sections of the wing.

As $A_n \sin n\varphi$ appears in the series for the lift distribution as well as in that for the angle-of-attack distribution, the former may equally well be expressed with the latter. Eliminating the last term of the series, affords

$$c_a t = \frac{\sin \varphi}{\xi + \frac{m}{8}} \left[\alpha + \frac{m-1}{8} A_1 \frac{\sin \varphi}{\sin \varphi} + \frac{m-2}{8} A_2 \frac{\sin 2\varphi}{\sin \varphi} \right.$$

$$\left. + \frac{m-3}{8} A_3 \frac{\sin 3\varphi}{\sin \varphi} + \dots \right]$$

$$= \frac{\sin \varphi}{\xi + \frac{m}{8}} \left[\alpha + \sum_{n=1}^{n=m} \frac{m-n}{8} A_n \frac{\sin n \varphi}{\sin \varphi} \right] \quad (7)$$

The subsequent calculations of the lift distribution are effected with special forms of this generalized series. The advantage in simplified calculation accruing from this remodeling rests not only in the fact that for a finite

arrangement of terms, only $m - 1$ coefficients need to be defined, but also that the fewness of coefficients in an approximate solution itself insures a high degree of accuracy in the approach.

The coefficients A_1 and A_2 are known through the choice of the operating condition; that is, A_1 defines the total lift and A_2 , the rolling moment. That is,

$$c_{a\text{total}} F = \int_{-1}^{+1} c_a t dy$$

$$\int_{-1}^{+1} c_a t dy = \frac{\pi}{2} A_1$$

$$A_1 = c_{a\text{total}} \frac{F}{\frac{\pi}{2}} \quad (8)$$

Further,

$$\frac{M_Q}{q} = \int_{-1}^{+1} c_a t y dy$$

$$\int_{-1}^{+1} c_a t y dy = -A_2 \frac{\pi}{4}$$

$$A_2 = -\frac{\frac{M_Q}{q}}{\frac{\pi}{4} q} \quad (9)$$

wherewith, other than α , the terms of the series constituting the major proportion of the bracketed term, are known. Interrupting the series at $m = 3$ or $m = 4$ for the first approach of the lift distribution, it affords a correction for $c_{a\text{total}}$ and the rolling moment, which in turn gives the correction for α_m and α_Q , thus neutralizing in part the error introduced through breaking off the series.

When defining the four elementary distributions, the series for the total distribution is split up into four parts and each is computed separately. The series for the normal distribution contains, for reasons of symmetry, only uneven terms; that is, A_1, A_3, A_5, A_7 , etc. and so does that for a zero distribution. In addition, it is $A_1 = 0$, because the total lift of a zero distribution was assumed equal to zero. The series for the normal roll distribution contains only even terms; that is, A_2, A_4, A_6 , etc., be-

cause the ordinates of the roll distribution for the left and right wings are inversely equivalent. The zero distribution in roll contains the even terms A_4, A_6 , etc., because according to definition the rolling moment and consequently $A_2 = 0$.

The total distribution is formed as the sum of the four elements:

$$c_a t_{\text{total}} = c_a t_{\text{normal}} + c_a t_{\Delta} + c_a t_Q + c_a t_{Q\Delta} \quad (10)$$

whence the series for the total distribution is the sum of the series of the elementary distributions; that is, the coefficients of equal $\sin n\varphi$ terms are additive. The total lift coefficients are hereinafter written with capital letters, namely: $A_1, A_2, A_3 \dots A_n$, those of the normal distribution with small letters a_1, a_3, a_5 to a_{2n+1} , or a_2, a_4, a_6 to a_{2n} , whereas those of the zero distribution, and zero distribution in roll are designated with a prime, thus: $a_3', a_5' \dots a'_{2n+1}$ and $a_4', a_6' \dots a'_{2n}$.

2. Normal Distribution

The determination of the series is explained in detail as follows. It is:

$$c_a t_{\text{normal}} = \frac{\sin \varphi}{\xi + \frac{2m+1}{8}}$$

$$\left[\alpha_m + \sum_{n=0}^{n=m} \frac{(2m+1) - (2n+1)}{8} a_{(2n+1)} \frac{\sin (2n+1)\varphi}{\sin \varphi} \right],$$

i.e., for $m = 3$, for example,

$$c_a t_{\text{normal}} = \frac{\sin \varphi}{\xi + \frac{7}{8}}$$

$$\left[\alpha_m + \frac{3}{4} a_1 + \frac{2}{4} a_3 \frac{\sin 3\varphi}{\sin \varphi} + \frac{1}{4} a_5 \frac{\sin 5\varphi}{\sin \varphi} \right]$$

Since the total lift in normal distribution and consequently a_1 is merely a scale factor, it is:

$$\frac{c_a t_{\text{normal}}}{a_1} = \frac{\sin \varphi}{\xi + \frac{2m+1}{8}} \left[\frac{\alpha_m}{a_1} + \frac{m}{4} + \frac{m-1}{4} \frac{a_3}{a_1} \frac{\sin 3 \varphi}{\sin \varphi} + \frac{m-2}{4} \frac{a_5}{a_1} \frac{\sin 5 \varphi}{\sin \varphi} + \dots \right] \quad (11)$$

We first approximate α_m/a_1 , because

$$\int_{-1}^{+1} \frac{c_a t_{\text{normal}}}{a_1} dy = \frac{\pi}{2}$$

Integration of the right side of the above equation gives as first integral:

$$J_1 = \left[\frac{\alpha_m}{a_1} + \frac{m}{4} \right] \int_{-1}^{+1} \frac{\sin \varphi}{\xi + \frac{2m+1}{8}} dy$$

The next integral is:

$$J_3 = \frac{m-1}{4} \frac{a_3}{a_1} \int_{-1}^{+1} \frac{\sin 3 \varphi}{\xi + \frac{2m+1}{8}} dy$$

and the other integrals:

$$J_{2n+1} = \frac{m-n}{4} \frac{a_{2n+1}}{a_1} \int_{-1}^{+1} \frac{\sin (2n+1) \varphi}{\xi + \frac{2m+1}{8}} dy$$

Bearing in mind that $\xi + \frac{2m+1}{8}$ differs only slightly from a mean value and that, besides, $\frac{m-n}{4} \frac{a_{2n+1}}{a_1}$ is small, it is found that the integrals from J_3 to J_{2n+1} are almost equal to zero, because

$$\int_{-1}^{+1} \sin (2n+1) \varphi dy = \int_0^{\pi} \sin (2n+1) \varphi \sin \varphi d\varphi = 0.$$

Accordingly, a close approach of α_m/a_1 is:

$$\frac{\alpha_m}{a_1} \approx \frac{\frac{\pi}{2}}{\int_{-1}^{+1} \frac{\sqrt{1-y^2}}{\xi + \frac{2m+1}{8}} dy} \quad (12)$$

The integral is best evaluated by graphical method, the numerical calculation being in most cases quite difficult. Computing this integral for the first three steps $m = 1, 2$, and 3 , the α_m/a_1 values are found to vary only a few percent, so that the mean value of these three steps is sufficiently accurate.

Even a mean value for ξ itself affords a satisfactory first approach. Then,

$$\frac{\alpha_m}{a_1} \sim \xi_{\text{mean}} + \frac{1}{8}$$

From the determination of α_m/a_1 follows the first approximation of the normal distribution, by breaking off the series expansion at the third term. Then,

$$\frac{c_a t_{\text{normal}}}{a_1} \textcircled{1} = \frac{\sqrt{1-y^2}}{\xi + \frac{3}{8}} \left[\frac{\alpha_m}{a_1} + \frac{1}{4} \right] \quad (13)$$

This first approach already affords information about the characteristic behavior of the normal distribution and is perfectly satisfactory for design calculations, the errors for smooth wing contours ranging between 3 and 5 percent.

To determine the higher coefficients, we merely compute that portion of the lift distribution formed as difference between the normal distribution and a pure elliptical distribution for equal total lift: Forcing a pure elliptical distribution on the wing by changing the angle of attack over the span, that is, writing

$$c_a t = a_1 \sin \varphi = a_1 \sqrt{1-y^2}$$

the angle-of-attack distribution is given through:

$$\alpha_{\text{elliptic}} = a_1 \left(\xi + \frac{1}{8} \right)$$

The difference between the constant α_m and the elliptical angle of attack represents the twist pertaining to the difference in lift distribution between normal and pure elliptical distribution. The differential grading is a zero distribution which changes with the total lift. It

is called elliptical-zero distribution. It is:

$$c_a t_{\text{normal}} = c_a t_{\text{elliptic}} + c_a t \Delta_{\text{elliptic}}$$

likewise

$$\alpha_m = a_1 \left(\xi + \frac{1}{8} \right) + \Delta_{\text{elliptic}}$$

To define the elliptical-zero distribution from the elliptical twist, we form:

$$\frac{\Delta_{\text{elliptic}}}{a_1} = \frac{\alpha_m}{a_1} - \left[\xi + \frac{1}{8} \right] \quad (14)$$

$$\begin{aligned} \frac{c_a t \Delta_{\text{elliptic}}}{a_1} = & \frac{\sin \varphi}{\xi + \frac{2m+1}{8}} \left[\frac{\Delta_{\text{elliptic}}}{a_1} + \frac{m-1}{4} \frac{a_3}{a_1} \frac{\sin 3 \varphi}{\sin \varphi} \right. \\ & + \frac{m-2}{4} \frac{a_5}{a_1} \frac{\sin 5 \varphi}{\sin \varphi} + \dots \\ & \left. + \frac{1}{4} \frac{a_{2m-1}}{a_1} \frac{\sin (2m-1) \varphi}{\sin \varphi} \right] \dots \quad (15) \end{aligned}$$

In first approach, we have:

$$\left[\frac{c_a t \Delta_{\text{elliptic}}}{a_1} \right]_{(1)} = \frac{\sqrt{1-y^2}}{\xi + \frac{3}{8}} \left[\frac{\Delta_{\text{elliptic}}}{a_1} \right] \quad (16)$$

Now, to what extent does this approach conform to actuality? Subtracting $\xi + \frac{1}{8}$ from the series for α_m , leaves

$$\begin{aligned} \frac{\Delta_{\text{elliptic}}}{a_1} = & \frac{a_3}{a_1} \frac{\sin 3 \varphi}{\sin \varphi} \left[\xi + \frac{3}{8} \right] + \frac{a_5}{a_1} \frac{\sin 5 \varphi}{\sin \varphi} \left[\xi + \frac{5}{8} \right] + \dots \\ & + \frac{a_{2m+1}}{a_1} \frac{\sin (2m+1) \varphi}{\sin \varphi} \left[\xi + \frac{2m+1}{8} \right] \end{aligned}$$

which reduces to

$$\left[\frac{\Delta_{\text{elliptic}}}{a_1} \right] \frac{\sin \varphi}{\xi + \frac{3}{8}} = \frac{a_3}{a_1} \sin 3 \varphi + \frac{a_5}{a_1} \sin 5 \varphi \left[\frac{\xi + \frac{5}{8}}{\xi + \frac{3}{8}} \right] \\ + \frac{a_7}{a_1} \sin 7 \varphi \left[\frac{\xi + \frac{7}{8}}{\xi + \frac{3}{8}} \right] + \dots \\ + \frac{a_{2m+1}}{a_1} \sin (2m+1) \varphi \left[\frac{\xi + \frac{2m+1}{8}}{\xi + \frac{3}{8}} \right]$$

The first approach differs from the true series for elliptical zero distribution only in the added quotient

$\left[\frac{\xi + \frac{2m+1}{8}}{\xi + \frac{3}{8}} \right]$. Therefore the first coefficients $\frac{a_3}{a_1}, \frac{a_5}{a_1}$, etc., may be closely approached from $\frac{\Delta_{\text{elliptic}}}{a_1} \frac{\sin \varphi}{\xi + \frac{3}{8}}$ when

defining the first Fourier series terms of this approach with the aid of harmonic analysis. That is, we write:

$$\frac{\Delta_{\text{elliptic}}}{a_1} \frac{\sin \varphi}{\xi + \frac{3}{8}} = c_3 \sin 3\varphi + c_5 \sin 5\varphi + \dots + c_n \sin n\varphi$$

The quotients $\frac{\xi + \frac{2m+1}{8}}{\xi + \frac{3}{8}}$ differing but little from

a mean value with respect to coordinate φ , the desired coefficients of the series may be approximated at

$$\frac{a_3}{a_1} \approx c_3$$

$$\frac{a_5}{a_1} \approx c_5 \left[\frac{\xi + \frac{3}{8}}{\xi + \frac{5}{8}} \right]_{\text{mean}}$$

$$\frac{a_7}{a_1} \approx c_7 \left[\frac{\xi + \frac{3}{8}}{\xi + \frac{7}{8}} \right]_{\text{mean}}$$

.....

A higher approach for $\frac{c_a t \Delta_{\text{elliptic}}}{a_1}$ is then obtained when inserting the thus-established coefficients into (15) such as:

$$\left[\frac{c_a t \Delta_{\text{elliptic}}}{a_1} \right]_{(3)} = \frac{\sin \varphi}{\xi + \frac{7}{8}} \left[\frac{\Delta_{\text{elliptic}}}{a_1} + \frac{1}{2} \frac{a_3}{a_1} \frac{\sin 3 \varphi}{\sin \varphi} + \frac{1}{4} \frac{a_5}{a_1} \frac{\sin 5 \varphi}{\sin \varphi} \right] \dots (17)$$

To ascertain the degree of accuracy of this approximation, we effect various reductions, so that:

$$\begin{aligned} \frac{\sin \varphi}{\xi + \frac{7}{8}} \left[\frac{\Delta_{\text{elliptic}}}{a_1} + \frac{1}{2} \frac{a_3}{a_1} \frac{\sin 3 \varphi}{\sin \varphi} + \frac{1}{4} \frac{a_5}{a_1} \frac{\sin 5 \varphi}{\sin \varphi} \right] = \\ = \frac{a_3}{a_1} \sin 3 \varphi + \frac{a_5}{a_1} \sin 5 \varphi + \frac{a_7}{a_1} \sin 7 \varphi + \frac{a_9}{a_1} \left[\frac{\xi + \frac{9}{8}}{\xi + \frac{7}{8}} \right] + \\ + \frac{a_{11}}{a_1} \sin 11 \varphi \left[\frac{\xi + \frac{11}{8}}{\xi + \frac{7}{8}} \right] + \dots \\ + \frac{a_{2m+1}}{a_1} \sin (2m+1) \varphi \left[\frac{\xi + \frac{2m+1}{8}}{\xi + \frac{7}{8}} \right] \end{aligned}$$

Again evaluating this approximation by means of harmonic analysis, yields the improved Fourier series coefficients d_3, d_5, d_7 , etc. After forming the mean values

for $\frac{\xi + \frac{2m+1}{8}}{\xi + \frac{7}{8}}$ the calculation gives the improved approach

of the desired coefficients of the elliptical zero distribution at:

$$\frac{a_3}{a_1} = d_3$$

$$\frac{a_5}{a_1} = d_5$$

$$\frac{a_7}{a_1} = d_7$$

$$\frac{a_9}{a_1} = d_9 \left[\frac{\xi + \frac{7}{8}}{\xi + \frac{9}{8}} \right]_{\text{mean}}$$

$$\frac{a_{11}}{a_1} = d_{11} \left[\frac{\xi + \frac{7}{8}}{\xi + \frac{11}{8}} \right]_{\text{mean}}$$

This method may be continued until the desired degree of accuracy is obtained.

It is seen that for engineering purposes in which accuracy need not exceed that of other aerodynamic principles, the coefficients may be interrupted with a_7/a_1 . In many cases even lower approaches are sufficient, when the wing contour differs only slightly from the elliptical.

3. Zero Distribution

That is, the distribution produced by equilateral twist. The angle-of-attack curve is given and must be separated into mean angle of attack and pure twist. In other words, the true angle of twist must be referred to the direction given through the zero-lift axis of the whole wing in place of the arbitrarily assumed body fixed reference plane. This separation is effected by bearing in mind that the desired zero distribution must comply with

$$\int_0^1 c_a t_{\Delta} dy = 0$$

The first approach is:

$$[c_a t_\Delta]_{(1)} = \frac{\sqrt{1 - y^2}}{\xi + \frac{3}{8}} \Delta \quad (18)$$

In view of $\Delta + \epsilon$ being given rather than Δ , whereby ϵ is the angle of zero lift direction and reference plane, we determine ϵ from the integral

$$\int_0^{+1} (\Delta + \epsilon) \frac{\sqrt{1 - y^2}}{\xi + \frac{3}{8}} dy = \epsilon \int_0^{+1} \frac{\sqrt{1 - y^2}}{\xi + \frac{3}{8}} dy$$

From the first approximation of the desired zero distribution, we subsequently determine in similar fashion as for the elliptical zero distribution, the first coefficients of the Fourier series through harmonic analysis.

With the zero distribution as:

$$c_a t_\Delta = a_3' \sin 3 \varphi + a_5' \sin 5 \varphi + \dots + a_{2m+1}' \sin (2m + 1) \varphi$$

it is:

$$\Delta = a_3' \frac{\sin 3 \varphi}{\sin \varphi} \left[\xi + \frac{3}{8} \right] + a_5' \frac{\sin 5 \varphi}{\sin \varphi} \left[\xi + \frac{5}{8} \right] + \dots + a_{2m+1}' \frac{\sin (2m + 1) \varphi}{\sin \varphi} \left[\xi + \frac{2m + 1}{8} \right]$$

so that the first approach of the zero distribution corresponds to the term:

$$\Delta \frac{\sin \varphi}{\xi + \frac{3}{8}} = a_3' \sin 3 \varphi + a_5' \sin 5 \varphi \left[\frac{\xi + \frac{5}{8}}{\xi + \frac{3}{8}} \right] + \dots + a_{2m+1}' \sin (2m + 1) \varphi \left[\frac{\xi + \frac{2m + 1}{8}}{\xi + \frac{3}{8}} \right]$$

With the use of the mean values of $\left[\frac{\xi + \frac{2m + 1}{8}}{\xi + \frac{3}{8}} \right]$, we

approximate the first coefficients and then form a higher approximation which may be improved as desired through further analysis. The calculation process being in principle the same as for the elliptical zero distribution, it needs no further explanation. In the closer approximations, ϵ must eventually be improved through control of the zero lift.

4. Normal Roll Distribution

Normal roll distribution produced, according to our definition, through straight, contrary wing twist; with α_{Q_m} as angle of roll, it is:

$$\alpha_{Q_m} = u y \quad (19)$$

The normal roll distribution is expressed in Fourier series with even terms:

$$c_a t_Q = a_2 \sin 2 \varphi + a_4 \sin 4 \varphi \\ + a_6 \sin 6 \varphi + \dots + a_{2m} \sin (2m) \varphi$$

The angle-of-attack curve is:

$$\alpha_{Q_m} = a_2 \frac{\sin 2 \varphi}{\sin \varphi} \left[\xi + \frac{2}{8} \right] + a_4 \frac{\sin 4 \varphi}{\sin \varphi} \left[\xi + \frac{4}{8} \right] + \dots \\ + a_{2m} \frac{\sin (2m) \varphi}{\sin \varphi} \left[\xi + \frac{2m}{8} \right],$$

whence:

$$c_a t_Q = \frac{\sin \varphi}{\xi + \frac{m}{4}} \left[\alpha_{Q_m} + \frac{m-1}{4} a_2 \frac{\sin 2 \varphi}{\sin \varphi} \right. \\ \left. + \frac{m-2}{4} a_4 \frac{\sin 4 \varphi}{\sin \varphi} + \dots \right. \\ \left. + \frac{m-n}{4} a_{2n} \frac{\sin (2n) \varphi}{\sin \varphi} \right] \dots \dots (20)$$

As the normal roll distribution changes with a_2 , we determine $\frac{c_a t_Q}{a_2}$, giving for the rolling moment:

$$\int_{-1}^{+1} \frac{c_a t_Q}{a_2} y dy = -\frac{\pi}{4}$$

Integration of the rolling moment from the above term for $c_a t_Q$ gives:

$$J_1 = \left[\frac{\alpha_{Qm}}{y} - \frac{m-1}{2} \right]_{-1}^{+1} \int \frac{y^2 \sqrt{1-y^2}}{\xi + \frac{m}{4}} dy$$

The next integral is:

$$J_2 = \frac{m-2}{4} \frac{a_4}{a_2} \int_{-1}^{+1} y \frac{\sin 4\varphi}{\xi + \frac{m}{4}} dy$$

and the other integrals are:

$$J_n = \frac{m-n}{4} \frac{a_{2n}}{a_2} \int_{-1}^{+1} y \frac{\sin (2n)\varphi}{\xi + \frac{m}{4}} dy$$

Then, $\int_{-1}^{+1} y \sin (2n)\varphi d\varphi = 0$ for $n > 1$.

Moreover, as $\xi + \frac{m}{4}$ differs very little from the mean value, the integrals J_2 to J_n are almost equal to zero. Thus, a serviceable approximation for the angle of roll obtains from:

$$\frac{\alpha_{Qm}/y}{a_2} \approx -\frac{\frac{\pi}{8}}{\int_0^{+1} y^2 \frac{\sqrt{1-y^2}}{\xi + \frac{m}{4}} dy} + \frac{m-1}{2} \quad (21)$$

The integral of the denominator is defined graphically. The mean value of $\frac{\alpha_{Qm}/y}{a_2}$ obtained for different m is within mathematical accuracy.

In first approximation the integral is determined by introducing a mean value for ξ . Then

$$\frac{\alpha_{Q_m}/y}{a_2} \sim -2 \left[\xi_{\text{mean}} + \frac{1}{4} \right]$$

Having determined $\frac{\alpha_{Q_m}/y}{a_2}$ the first approximation of the normal roll distribution obtains at:

$$\left[\frac{c_a t_Q}{a_2} \right] \textcircled{1} = \left[\frac{\alpha_{Q_m}/y}{a_2} - \frac{1}{2} \right] \frac{y \sqrt{1 - y^2}}{\xi + \frac{1}{2}} \quad (22)$$

The other coefficients must be determined if the higher approaches are derivated. We again separate the elliptical roll distribution, given through

$$\frac{c_a t_{Q_{\text{elliptic}}}}{a_2} = \sin 2 \varphi$$

from the rest, a zero distribution in roll, which changes as a_2 . It is:

$$\frac{(c_a t_Q) \Delta}{a_2} = \frac{c_a t_Q}{a_2} - \sin 2 \varphi \quad (23)$$

The angle-of-attack distribution pertinent to $(c_a t_Q) \Delta$, is:

$$\frac{\Delta_Q}{a_2} = \frac{\alpha_{Q_m}}{a_2} - \frac{\sin 2 \varphi}{\sin \varphi} \left[\xi + \frac{1}{4} \right]$$

or, when using $\frac{\Delta_Q/y}{a_2}$ for forming the other approximations:

$$\frac{\Delta_Q/y}{a_2} = \frac{\alpha_{Q_m}/y}{a_2} + 2 \left[\xi + \frac{1}{4} \right] \dots \dots \dots (24)$$

On the other hand, it must:

$$\begin{aligned} \frac{\Delta_Q/y}{a_2} = & -2 \frac{a_4}{a_2} \frac{\sin 4 \varphi}{\sin 2 \varphi} \left[\xi + \frac{1}{2} \right] \\ & - 2 \frac{a_6}{a_2} \frac{\sin 6 \varphi}{\sin 2 \varphi} \left[\xi + \frac{3}{4} \right] - \dots \\ & - 2 \frac{a_{2n}}{a_2} \frac{\sin (2n) \varphi}{\sin 2 \varphi} \left[\xi + \frac{n}{4} \right] . \end{aligned}$$

The first approximation of the zero distribution in roll being defined as

$$\left[\frac{(c_a t_Q) \Delta}{a_2} \right]_{(1)} = \left[\frac{\Delta_Q/y}{a_2} \right] \frac{y \sqrt{1-y^2}}{\xi + \frac{1}{2}} \quad (25)$$

the coefficients are computed from:

$$\begin{aligned} \frac{\Delta_Q/y}{a_2} \frac{y \sqrt{1-y^2}}{\xi + \frac{1}{2}} &= \frac{a_4}{a_2} \sin 4 \varphi \\ &+ \frac{a_6}{a_2} \sin 6 \varphi \left[\frac{\xi + \frac{3}{4}}{\xi + \frac{1}{2}} \right] + \dots \\ &+ \frac{a_{2n}}{a_2} \sin (2n) \varphi \left[\frac{\xi + \frac{n}{4}}{\xi + \frac{1}{2}} \right] \end{aligned}$$

by again averaging $\left[\frac{\xi + \frac{m}{4}}{\xi + \frac{1}{2}} \right]$ and then correcting the coef-

ficients obtained from harmonic analysis. The higher approach for the desired elliptical zero distribution in roll, then, is:

$$\begin{aligned} \left[\frac{(c_a t_Q) \Delta}{a_2} \right]_{(m)} &= \frac{y \sqrt{1-y^2}}{\xi + \frac{m}{4}} \left[\frac{\Delta_Q/y}{a_2} \right. \\ &\quad - \frac{(m-2)}{2} \frac{a_4}{a_2} \frac{\sin 4 \varphi}{\sin 2 \varphi} - \dots \\ &\quad \left. - \frac{(m-n)}{2} \frac{a_{2n}}{a_2} \frac{\sin (2n) \varphi}{\sin 2 \varphi} \right] \quad (26) \end{aligned}$$

and the coefficients in m^{th} approach are determined through:

$$\begin{aligned}
& \frac{y \sqrt{1-y^2}}{\xi + \frac{m}{4}} \left[\frac{\Delta_Q/y}{a_2} - \frac{(m-2)}{2} \frac{a_4}{a_2} \frac{\sin 4\varphi}{\sin 2\varphi} \right. \\
& \quad \left. - \frac{(m-n)}{2} \frac{a_{2n}}{a_2} \frac{\sin (2n)\varphi}{\sin 2\varphi} \right] = \\
& = \sum_{n=2}^{n=m} \frac{a_{2n}}{a_2} \sin (2n)\varphi + \sum_{n=m+1}^{n=\infty} \frac{a_{2n}}{a_2} \sin (2n)\varphi \left[\frac{\xi + \frac{n}{4}}{\xi + \frac{m}{4}} \right]
\end{aligned}$$

To illustrate, for $m = 4$, it is:

$$\begin{aligned}
& \frac{y \sqrt{1-y^2}}{\xi + \frac{4}{4}} \left[\frac{\Delta_Q/y}{a_2} - \frac{2}{2} \frac{a_4}{a_2} \frac{\sin 4\varphi}{\sin 2\varphi} - \frac{1}{2} \frac{a_6}{a_2} \frac{\sin 6\varphi}{\sin 2\varphi} \right] = \\
& = \frac{a_4}{a_2} \sin 4\varphi + \frac{a_6}{a_2} \sin 6\varphi + \frac{a_8}{a_2} \sin 8\varphi \\
& \quad + \frac{a_{10}}{a_2} \sin 10\varphi \left[\frac{\xi + \frac{5}{4}}{\xi + \frac{4}{4}} \right] \\
& \quad + \frac{a_{12}}{a_2} \sin 12\varphi \left[\frac{\xi + \frac{6}{4}}{\xi + \frac{4}{4}} \right] \\
& \quad + \frac{a_{14}}{a_2} \sin 14\varphi \left[\frac{\xi + \frac{7}{4}}{\xi + \frac{4}{4}} \right] + \dots
\end{aligned}$$

The process is continued until the desired accuracy is obtained.

5. Zero Distribution in Roll

The roll distribution due to aileron deflection is computed on the basis of a distribution composed of a normal roll and zero distribution in roll. As a result, the additional angle-of-attack curve due to aileron deflection must be separated into the normal proportion and a twist

proportion (fig. 5). As the additive zero angle of attack due to aileron deflection is proportional to the angle of aileron deflection, we plot the ratio of both angles, that is:

$$\frac{\alpha_{oQ}}{\beta} = \epsilon$$

$$\text{whereas,} \quad \epsilon = \alpha_{Q_m} + \Delta Q \quad (27)$$

$$\text{or} \quad \frac{\epsilon}{y} = \frac{\alpha_{Q_m}}{y} + \frac{\Delta Q}{y} \quad (27a)$$

Thus the problem reduces to finding the constant quota $\frac{\alpha_{Q_m}}{y}$ for a given $\frac{\epsilon}{y}$. This separation is effected on the basis that the rolling moment due to ϵ equals the rolling moment due to α_{Q_m} .

In first approach, we have:

$$(c_a t)_Q = \left[\frac{\epsilon}{y} - \frac{a_2}{2} \right] \frac{y \sqrt{1 - y^2}}{\xi + \frac{1}{2}} \quad (28)$$

with a rolling moment:

$$\int_{-1}^{+1} (c_a t)_Q y dy = \int_{-1}^{+1} \frac{\epsilon}{y} \frac{y^2 \sqrt{1 - y^2}}{\xi + \frac{1}{2}} dy - \frac{a_2}{2} \int_{-1}^{+1} \frac{y^2 \sqrt{1 - y^2}}{\xi + \frac{1}{2}} dy$$

Contrariwise, as

$$\int_{-1}^{+1} (c_a t)_Q y dy = -a_2 \frac{\pi}{4}$$

a_2 is defined with

$$a_2 = \frac{2 \int_0^{+1} \frac{\epsilon}{y} \frac{y^2 \sqrt{1 - y^2}}{\xi + \frac{1}{2}} dy}{\int_0^{+1} \frac{y^2 \sqrt{1 - y^2}}{\xi + \frac{1}{2}} dy - \frac{\pi}{4}} \quad (29)$$

Having, for computing the normal roll distribution, defined the mean angle of roll referred to a_2 , it affords α_{Qm}/y , so that the twist follows from the differentiation of angle roll Δ_Q/y .

Formed as before, the first approach of the desired zero distribution in roll then is:

$$[(c_a t)_Q]_{\Delta(1)} = \frac{\Delta_Q}{y} \frac{y \sqrt{1 - y^2}}{\xi + \frac{1}{2}} \quad (30)$$

from which the higher coefficients are obtained by harmonic analysis; a_2 being determined as first approach, the zero rolling moment condition must be controlled for higher approaches and further corrections effected on α_{Qm}/y and Δ_Q/y .

V. GRAPHICAL ANALOGY

The mathematical lift distribution may also be replaced by a graphical method having the advantage of clearness as well as lucidity of the mathematical treatment of the problem. The principle is, briefly, as follows:

Plotting c_{at}/a_1 against angle of attack α/a_1 affords for such lift distributions as formed from the first two terms a_1/a_1 and a_3/a_1 the relationship portrayed in figure 6. The c_{at}/a_1 values pertaining to certain points on the span lie on straight lines passing through point $\frac{\alpha_m}{a_1} = -\frac{1}{4}$, $\frac{c_a t}{a_1} = 0$. For in view of

$$\frac{c_a t}{a_1} = \sin \varphi + \frac{a_3}{a_1} \sin 3\varphi$$

$$\frac{\alpha}{a_1} = \left[\xi + \frac{1}{8} \right] + \frac{a_3}{a_1} \frac{\sin 3\varphi}{\sin \varphi} \left[\xi + \frac{3}{8} \right]$$

it is for $\frac{c_a t}{a_1} = 0$,

$$\frac{\alpha}{a_1} = -\frac{1}{4}.$$

With this a desired lift distribution can be graphically obtained in first approach, as follows:

Find, for a given chord distribution and angle of attack, the lift distribution giving in first approach a certain total lift. In the spatial system of coordinates $\frac{\alpha}{a_1}$, $\frac{c_a t}{a_1}$, and y (fig. 7), we plot the lift distribution conformably to pure elliptical distribution

$$\left(\frac{c_a t}{a_1} = \sqrt{1 - y^2}; \frac{\alpha}{a_1} = \xi + \frac{1}{8} \right).$$

Then we draw the straight $\frac{c}{a_1} = -\frac{1}{4}$ and establish an area through the elliptical lift distribution and this straight line, in which all lift distributions are contained which give the total lift

$$\int_0^{+1} \frac{c_a t}{a_1} dy = + \frac{\pi}{4}$$

and can be formed through the first two terms of the Fourier series. Presuming that the exactly possible lift distribution in first approach can be interpolated between the exact lift distributions, the problem consists in finding that curve on the area whose projection marks off the given angle of attack from the α/a_1 to y plane, and whose content in $c_a t/a_1$ to y plane is π . This construction is illustrated in figure 7 for a normal distribution with constant angle of attack. It is readily proved that this first approach of the lift distribution is conformably to the derived first approach of the lift distribution:

$$\frac{c_a t}{a_1} = \left[\frac{\alpha}{a_1} + \frac{1}{4} \right] \frac{\sqrt{1 - y^2}}{\xi + \frac{3}{8}}$$

Closer approaches can be determined in like manner in the 3-coordinate system by defining, say, coefficient a_3/a_1 for the second approach (harmonic analysis, first approach). Then, however, the track for the area of coordinated lift distributions assumes the function:

$$\left[\frac{\alpha}{a_1} \right]_{\text{track}} = - \left[\frac{1}{2} + \frac{1}{4} \frac{a_3}{a_1} (4y^2 - 1) \right]$$

as illustrated in figure 8.

Further approaches may be formed in analogous manner after the corresponding coefficients have been obtained from analysis. Obviously, such a continuation of graphical determination is of no practical significance because, in most cases, the plotting accuracy is insufficient to present the still possible improvements of the solution. The method is, above all, suitable for intelligibly showing the method and for effectuating graphically the determination of the first approach.

There is still another graphical method which is even more simple. (See fig. 9.) On three planes intersecting in a straight line are the functions $f(y)$, $F(y)$, and $\varphi(y)$ together with the stipulation that the points of the three functions pertaining to equal y values, lie on a straight line. Then, we have, according to figure 9:

$$F(y) = \frac{\sin(\alpha + \beta)}{\cos \beta} \frac{f(y) \varphi(y)}{\varphi(y) + f(y) \frac{\cos \alpha}{\cos \beta}}$$

On the other hand, forming the relation for the first approach of the lift distribution affords:

$$\frac{c_a t}{a_1} = \left[\frac{\alpha}{a_1} + \frac{1}{4} \right] \frac{2\pi \eta t \sqrt{1 - y^2}}{\sqrt{1 - y^2} + 2\pi \eta t \frac{3}{8}}$$

which for

$$2\pi \eta t = f(y)$$

$$\sqrt{1 - y^2} = \varphi(y)$$

$$\frac{\cos \alpha}{\cos \beta} = \frac{3}{8}$$

$$\frac{\sin(\alpha + \beta)}{\cos \beta} = \left[\frac{\alpha}{a_1} + \frac{1}{4} \right]$$

gives

$$\frac{c_a t}{a_1} = F(y).$$

This form affords higher approaches when inserting the relations between length ordinate y and angles α and β as new variables. This method can be readily developed so that a mechanical drawing instrument can be used for defining the lift distribution.

The above-cited derivation for the graphical determination of the lift distribution applies only to normal and zero distribution, although there is a similar method for the roll distribution. As hereby the rolling moment is held constant rather than the roll distribution itself, the rolling-moment distribution is determined through projection. Figure 10 shows the construction for defining in first approach a normal roll distribution, where it is readily seen that:

$$\frac{c_a}{a_2} t_Q y = \left[\frac{\alpha_Q/y}{a_2} - \frac{1}{2} \right] \frac{y^2 \sqrt{1 - y^2}}{\xi + \frac{1}{2}}$$

The track is here also a straight line through $\frac{\alpha_Q/y}{a_2} = + \frac{1}{2}$. Further approaches are deduced in the same manner as above, by defining the higher track functions.

For a wing whose chord or angle-of-attack curve reveals corners so that dt/dy or $d\alpha/dy$ becomes unsteady at these points, the method repeated here yields corners in the lift distribution in the lower approaches. Since, in this case $d(c_a t)/dy$ also becomes unsteady in these points, the integral for α_i gives the absolute term ∞ ; that is, the approximation at such points gives a mathematically wrong result. Even so, this has no effect on the accuracy of the final result.

To avoid such discontinuity points, the corners in chord or angle-of-attack distribution can be rounded off, as is, in fact, common practice in airplane design.

VI. EXAMPLE

The wing is a normal wing of conventional design as seen from the semiwing illustrated in figure 11. The pertinent data are appended in table I. The efficiency of the lift increase is $\eta = 0.89$, corresponding to a 12 to 14

percent chord thickness. Integral $\int_0^1 \frac{\sqrt{1 - y^2}}{\xi + \frac{2m + 1}{8}} dy$ was

graphically evaluated for $m = 1, 2$, and 3 from the following table, after which α_m/a_1 was defined according to (12). The compilation appended in the table reveals the discrepancies so small that the mean angle of attack can be determined within an accuracy of less than 1 percent.

TABLE I. $F = 0.4894$ $\Lambda = 8.173$

y	$b/2=1$ t	ξ	$\xi + \frac{1}{8}$	$\frac{\Delta}{a_{1ell}}$	$\left[\frac{c_a^t \Delta_{ell}}{a_1} \right] \textcircled{1}$	$\left[\frac{c_a^t \Delta_{ell}}{a_1} \right] \textcircled{2}$	$\left[\frac{c_a^t \Delta_{ell}}{a_1} \right] \textcircled{3}$	$\left[\frac{c_a^t \Delta_{ell}}{a_1} \right] \textcircled{4}$
0.00	0.3200	0.5534	0.6834	+0.0161	+0.0173	+0.0121	+0.0145	+0.0156
.10	.3200	.5553	.6808	+0.0137	+0.0200	+0.0144	+0.0157	+0.0161
.20	.3200	.5471	.6721	+0.0274	+0.0291	+0.0216	+0.0200	+0.0186
.30	.2975	.5730	.6980	+0.0015	+0.0015	+0.0003	-0.0001	-0.0003
.40	.2750	.5953	.7203	-0.0208	-0.0196	-0.0161	-0.0161	-0.0160
.50	.2525	.6129	.7379	-0.0384	-0.0337	-0.0269	-0.0269	-0.0266
.60	.2300	.6212	.7462	-0.0467	-0.0375	-0.0295	-0.0295	-0.0293
.70	.2075	.6150	.7400	-0.0405	-0.0292	-0.0224	-0.0219	-0.0216
.80	.1850	.5795	.7045	-0.0050	-0.0031	-0.0011	-0.0005	-0.0001
.90	.1600	.4869	.6119	+0.0876	+0.0443	+0.0359	+0.0348	+0.0341
.95	.1235	.4520	.5770	+0.1225	+0.0463	+0.0368	+0.0367	+0.0370
.975	.0910	.4365	.5615	+0.1380	+0.0378	+0.0299	+0.0303	+0.0309
1.00	.0000	.4270	.5520	+0.1475	\pm .0000	\pm .0000	\pm .0000	\pm .0000

TABLE II

m	$\int_0^1 \frac{\sqrt{1-y^2}}{\xi + \frac{2m+1}{8}} \alpha y$	$\frac{\alpha_m}{a_1}$
1	0.8272	0.6995
2	.6543	.7000
3	.5420	.6990

$$\left[\frac{\alpha_m}{a_1} \right]_{\text{mean}} = .6995$$

With 0.695, the value of α_m/a_1 defined as the 0th approach over the mean value of ξ , is also sufficiently accurate for rough calculations. A control check for the lift increase of the total wing must, of course, give a flatter rise than that of an elliptical wing with equal Λ and η . The following tabulation gives the established values.

$$\left[\frac{\partial c_a}{\partial \alpha} \right]_{\text{ell}} = 2\pi \eta \frac{\Lambda}{\Lambda + 2\eta} = 4.595$$

$$\left[\frac{\partial c_a}{\partial \alpha} \right]_{\text{eff}} = \frac{\pi}{2 F \frac{\alpha_m}{a_1}} = 4.588$$

The plan form of the wing being not much unlike that of an elliptical contour, the reduction in lift increase is negligible.

Following the determination of the elliptical $\frac{\Lambda}{a_1}_{\text{ell}}$ the four stages of approach of the elliptical zero distribution were calculated. The figures are included in table I. The determination of the coefficients from the harmonic analysis of the individual approaches is readily seen in table III in comparison with the coefficients obtained from the individual stages.

TABLE III

m	$\frac{a_3}{a_1}$	$\frac{a_5}{a_1}$	$\frac{a_7}{a_1}$
1	+0.0069	+0.0312	+0.0022
2	+0.0063	+0.0314	+0.0020
3	+0.0062	+0.0312	+0.0025

TABLE IV. Normal Distribution for $c_{atotal} = 1.00$

y	$c_{at}^{(1)}$	$c_{at}^{(2)}$	$c_{at}^{(3)}$	$c_{at}^{(4)}$	$c_a^{(1)}$	$c_a^{(2)}$	$c_a^{(3)}$	$c_a^{(4)}$
0.00	0.3171	0.3155	0.3164	0.3166	0.9910	0.9860	0.9889	0.9896
.10	.3165	.3147	.3151	.3153	.9890	.9833	.9850	.9853
.20	.3146	.3122	.3116	.3113	.9833	.9759	.9740	.9730
.30	.2979	.2975	.2974	.2973	1.0010	.9998	.9992	.9990
.40	.2796	.2808	.2808	.2808	1.0167	1.0210	1.0210	1.0210
.50	.2596	.2616	.2616	.2618	1.0292	1.0361	1.0361	1.0367
.60	.2378	.2403	.2403	.2403	1.0340	1.0446	1.0446	1.0446
.70	.2135	.2156	.2159	.2160	1.0289	1.0394	1.0403	1.0409
.80	.1861	.1867	.1870	.1870	1.0060	1.0092	1.0094	1.0094
.90	.1498	.1472	.1468	.1466	.9361	.9200	.9173	.9161
.95	.1118	.1089	.1088	.1089	.9052	.8818	.8810	.8818
.975	.0811	.0786	.0788	.0790	.8911	.8639	.8660	.8681
1.00	.0000	.0000	.0000	.0000	.8817	.8530	.8572	.8620

TABLE V

y	$2 \left[\frac{5}{4} - \frac{1}{4} \right]$	$\left[\frac{\Delta_Q/y}{a_2} \right]$	$\left[\frac{c_{atQ\Delta}}{a_2} \right] \textcircled{1}$	$\left[\frac{c_{atQ\Delta}}{a_2} \right] \textcircled{2}$	$\left[\frac{c_{atQ\Delta}}{a_2} \right] \textcircled{3}$	$\left[\frac{c_{atQ}}{a_2} \right] \textcircled{3}$
0.00	1.6168	-0.0244	± 0.0000	± 0.0000	± 0.0000	± 0.0000
.10	1.6116	-.0296	-.0028	+.0008	+.0004	+.1994
.20	1.5942	-.0470	-.0088	-.0014	-.0011	+.3908
.30	1.6460	+.0048	+.0013	+.0083	+.0083	+.5807
.40	1.6906	+.0494	+.0165	+.0210	+.0211	+.7543
.50	1.7258	+.0846	+.0329	+.0333	+.0336	+.8996
.60	1.7424	+.1012	+.0433	+.0394	+.0398	+.9998
.70	1.7300	+.0888	+.0398	+.0328	+.0326	+1.0324
.80	1.6590	+.0178	+.0079	+.0023	+.0017	+.9617
.90	1.4738	-.1674	-.0665	-.0611	-.0595	+.7251
.95	1.4040	-.2372	-.0739	-.0666	-.0667	+.5266
.975	1.3730	-.2682	-.0620	-.0557	-.0566	+.3767
1.00	1.3540	-.2872	± 0.0000	± 0.0000	± 0.0000	± 0.0000

TABLE VI

y	ϵ	$\frac{\epsilon}{y}$	$\Delta Q/y$	$[c_a^t Q]_{\Delta}$ ①	$[c_a^t Q]_{\Delta}$ ②	$[c_a^t Q]_{\Delta}$ ③	$[c_a^t Q]_{\Delta}$ ④
0.00	0.000	0.0000	+0.7500	± 0.0000	± 0.0000	± 0.0000	± 0.0000
.10	.000	.0000	+ .7500	+ .0707	+ .0597	+ .0569	+ .0605
.20	.000	.0000	+ .7500	+ .1404	+ .1160	+ .1106	+ .1134
.30	.000	.0000	+ .7500	+ .2000	+ .1637	+ .1545	+ .1501
.40	.480	1.2000	{ + .7500 - .4500	+ .2510 - .1506	+ .2108 - .1162	+ .1882 - .0876	+ .1713 - .0672
.50	.508	1.0160	- .2660	- .1035	- .0790	- .0620	- .0555
.60	.536	.8933	- .1433	- .0613	- .0468	- .0416	- .0449
.70	.571	.8157	- .0657	- .0295	- .0238	- .0275	- .0321
.80	.611	.7638	- .0138	- .0061	- .0034	- .0153	- .0122
.90	.660	.7333	+ .0167	+ .0066	- .0015	- .0036	- .0011
.95	.743	.7822	- .0322	- .0100	- .0148	- .0102	- .0095
.975	.850	.8718	- .1218	- .0282	- .0279	- .0196	- .0195
1.00	1.000	1.0000	- .2500	± 0.0000	± 0.0000	± 0.0000	± 0.0000

The existing discrepancies are within mathematical accuracy and the coefficients obtained from averaging

$$\frac{\xi + \frac{2m+1}{8}}{\xi + \frac{3}{8}}, \text{ etc. themselves prove sufficiently exact. The}$$

course of the essential functions is shown in figure 12.

In order to bring out the accuracy of the calculation for determining a conventional flight altitude, the four approaches of the normal distributions were reduced to a flight condition of $c_{atotal} = 1.0$ and the lift coefficient defined therefrom. The figures are appended in table IV and illustrated in figure 13. It is clearly seen that even the first approach reproduces the characteristic form of the lift distribution quite implicitly. The improvements through stages of approach are so minute that graphically it is barely possible to give the fourth approach, and in the c_a coefficient itself the discrepancies scarcely exceed the errors involved in the measurement of those coefficients.

The calculation of a zero distribution is omitted because it is practically identical with the elliptical zero distribution, and we proceed to the roll distribution due to aileron deflection. To this end we first establish the normal roll distribution of the wing. The figures are compiled in table V. It is seen that the values in third approach are already sufficiently convergent. For simplicity's sake $\frac{c_{m/y}}{a_2}$ was merely computed for $m = 2$, according to (21) at -1.6412 (fig. 14).

This brings us to the aileron effect itself. The shape and size of the aileron is seen from figure 11; the calculation process is given in table VI. The aileron effect for varying percent of aileron chord was determined according to the curve shown in figure 15. The curve was defined from experimental data. The reading gave $\epsilon = \frac{\partial \Delta \alpha_0}{\partial \beta}$ for different points of the aileron, while the forming of the integral:

$$\int_0^1 \frac{\epsilon}{y} \frac{y^2 \sqrt{1-y^2}}{\xi + \frac{1}{2}} dy$$

gave the value of a_2 which renders possible the separation of the additive aileron setting angle into normal angle of roll and zero angle in roll Δ_0/y . From the latter the pertinent zero distribution in roll was then obtained in progressive approaches. They are included in table VI. Figure 16 illustrates the course of the essential functions. As anticipated, the convergence is not as appreciable, due chiefly to the unsteady jump in angle of attack at $y = 0.40$. For this reason it is important to so combine the mathematical result that the differences in the approximate determination of the aileron lift distribution are presented with reference to an ordinary aileron setting angle. Choosing $\beta = +10^\circ$ affords the figures for the roll distribution with 10° aileron setting, given for the first and fourth approach in table VII.

TABLE VII

y	c_{atQ} ①	c_{atQ} ④	c_a ①	c_a ④
0.00	± 0.0000	± 0.0000	± 0.000	± 0.000
.10	+0.0037	+0.0053	+0.012	+0.017
.20	+0.0075	+0.0116	+0.024	+0.035
.30	+0.0107	+0.0188	+0.036	+0.063
.40	{ +0.0134 +0.0834	+0.0269 +0.0685	+0.049 +0.303	+0.098 +0.240
.50	+0.0848	+0.0763	+0.336	+0.302
.60	+0.0838	+0.0812	+0.364	+0.353
.70	+0.0818	+0.0838	+0.394	+0.404
.80	+0.0771	+0.0786	+0.417	+0.425
.90	+0.0667	+0.0675	+0.417	+0.422
.95	+0.0549	+0.0543	+0.445	+0.440
.975	+0.0444	+0.0425	+0.488	+0.437
1.000	± 0.0000	± 0.0000		

The corresponding lift coefficients were determined therefrom. The course is seen from figure 17. The errors in lift coefficient are minute. To remain within the limits of accuracy of the measured aerodynamic coefficients, the second approach for computing the aileron effect would have sufficed.

In conclusion, it is pointed out that an arithmetical scheme for a complete calculation of lift distribution conformable to the depicted method is being published elsewhere.

VII. COMPARATIVE CALCULATIONS

In order to prove the practicability of the analyzed method, we made some comparisons with other known methods. The normal and zero distribution of a wing, computed by Lotz's method, was published by S. Hueber (reference 3). This is the wing of the "Flavag III" performance glider, the computed data on lift distribution having been kindly placed at our disposal by the A.V.A., Göttingen. Figure 18 compares the mathematical results for a normal distribution of the wing; figure 19, the effect of twist. (See figs. 5 and 6 of reference 3.) Admittedly, in this case, it is not a pure zero distribution because the total lift is negative. But, to make possible the use of the numerical values, a conversion was forgone. The comparison reveals more pronounced discrepancies between Lotz's figures and our own,* but only negligible differences for the twisting effect.

In order to explain the difference between the two calculations, we defined the induced α_i for both lift distributions with our own method. The results are tabulated in

TABLE VIII

y	$\alpha_{i\text{Lotz}}$	$\alpha_{i\text{Lippisch}}$
0.00	+0.0301	+0.0316
.30	+0.0355	+0.0361
.50	+0.0404	+0.0378
.70	+0.0395	+0.0388

The determination of α_i involves a graphical integration of the curves obtained from the calculation rather than an analytical method.

Next, in order to ascertain the quality of the approach of the pertinent lift distributions, the local angle of at-

*This example was carried through according to the model calculation of the report: Aerodynamic Calculation of Airplane Wings (DFS). This model corresponds to the second approach of our method.

tack is determined at

$$\alpha = \frac{c_a t}{2\pi \eta t} + \alpha_i$$

from $c_a t$, α_i and the given chord t .

Inasmuch as the normal distribution proceeds from an angle of attack constant for all points of the span, the computed α values must agree if the solution is correct.

Table IX shows the results for both methods, while figure 20 illustrates the inversely determined α distribution for both methods.

TABLE IX

y	α_{Lotz}	α_{Lippisch}
0.00	+0.2499	+0.2559
.30	+.2526	+.2564
.50	+.2600	+.2565
.70	+.2593	+.2546

This table discloses that the discrepancies are due to the fact that Lotz's method resulted in more inaccurate figures. Another method of comparison would be to determine the chord distribution from the mathematically defined mean angle of attack. The mean angle of attack of Hueber's calculation being unknown, the comparison was effected on the basis of the mean angle of attack of our own calculation.

TABLE X

y	t_{actual}	t_{Lotz}	t_{Lippisch}
0.00	0.2333	0.2269	0.2333
.30	.2333	.2298	.2338
.50	.2185	.2225	.2187
.70	.1888	.1917	.1877

The results are collected in table X and figure 21.

Summing up, it is seen that the lift distribution defined from the second approach of our method already yields very practical results, whereby the determination of the

second approach - mathematically - can be carried through very quickly.

Figure 22 illustrates the results of a comparison with Glauert's method for the zero distribution of a tapered wing with twist, once, according to Glauert's four-point method, and then according to the model for determining the second approach of the previously cited publication. The accord is within accuracy of calculation.

The check of the calculation method for determining the aileron effect was made on an example cited in I. Lotz's report, Part III (reference 4), in which the coefficients of rolling moment, induced yawing moment, etc., are determined for a rectangular wing with variable aileron length. According to Lotz, the coefficient of rolling moment ζ , is defined as

$$\zeta = \frac{c_{m_q}}{\pi \frac{c_1}{8} \frac{b}{t_0} \alpha_q}$$

wherein

$$c_{m_q} = \frac{M_Q}{q F t}$$

and

$$c_1 = \frac{1}{2} \left[\frac{\partial c_a}{\partial \alpha} \right]_{b \rightarrow \infty} = \pi \eta$$

Lotz's coefficient of induced yawing moment ξ is:

$$\xi = \frac{c_{m_s}}{\pi \frac{c_1^2}{16} \alpha_g \alpha_q}$$

with

$$c_{m_s} = \frac{M_s}{q F t}$$

(α_g = total angle of attack.)

We determined the first approach of the aileron-lift distribution for Lotz's constants at

$$c_a t_Q = \frac{y \sqrt{1 - y^2}}{\xi + \frac{1}{2}} \left[\frac{\alpha_Q}{y} - \frac{a_2}{2} \right]$$

and the normal distribution additive for defining the yawing moment from the first approach at

$$c_{a \ t} = \frac{\sqrt{1 - y^2}}{\xi + \frac{3}{8}} \left[\alpha_g + \frac{a_1}{4} \right]$$

The calculation of the distributions conformably to these relations having been explained elsewhere, requires no further elucidation.

Our coefficients were obtained through graphical integration of the particular function. The results of the comparison are shown in figures 23 and 24 for the rolling and yawing moments of the wing with 1:5 aspect ratio.

These graphs were taken from Lotz's report (figs. 2 and 3) and our computed points added. Even the first approach is already seen to be in close agreement with Lotz's curve, while the dashed curve of the determination according to Wieselsberger-Glauert, discloses greater discrepancies.

The local errors of the first approach of the method elucidated here, have no effect on the results of calculations analyzing the effect of the wing as a whole. So that even the investigation of aileron effect of varying contours and aileron forms should afford practical results with the simple relations of the first approach of our method.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

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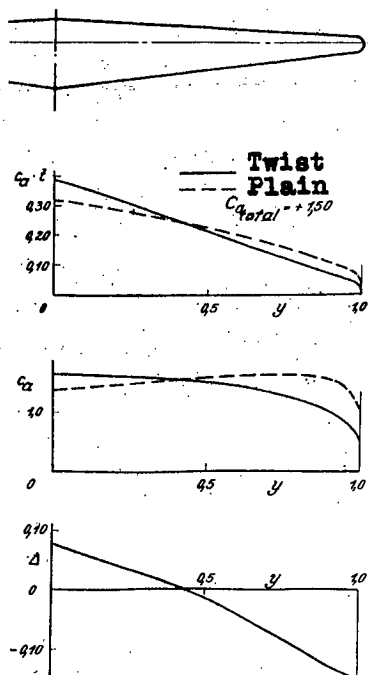


Figure 1.- Lift curve C_L of a tapered wing with and without twist.

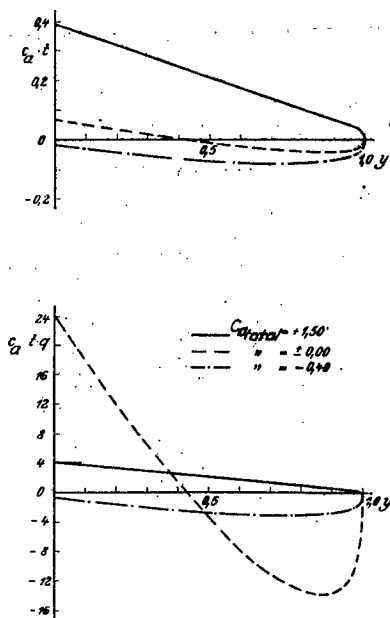


Figure 2.- Load distribution of a tapered wing with twist.

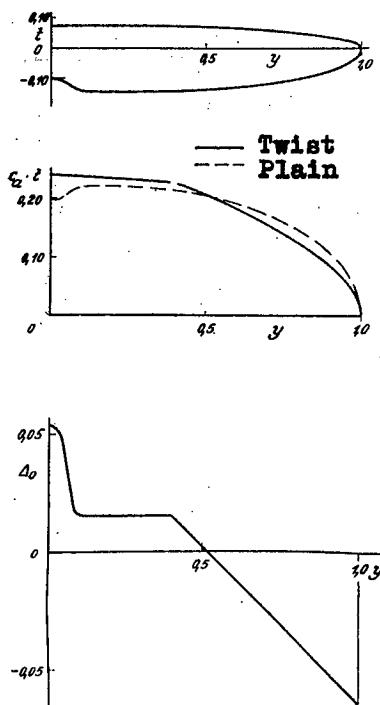


Figure 3.- Wing with cutaway.

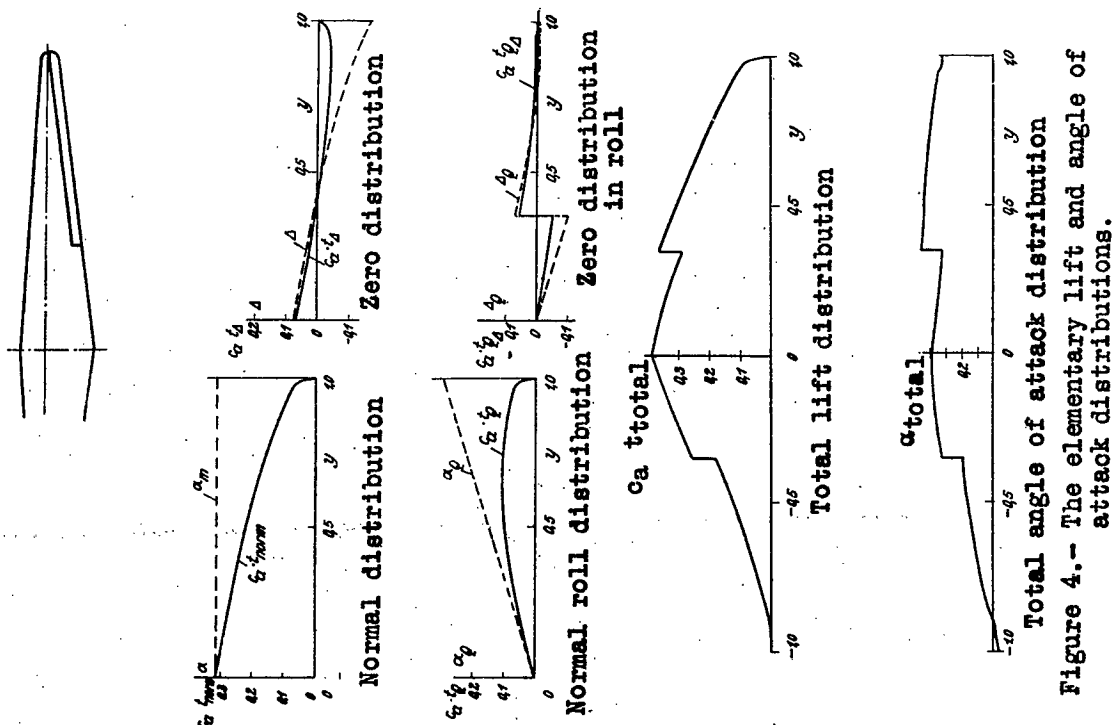


Figure 4.- The elementary lift and angle of attack distributions.

Figure 8.- Graphical determination of the second approach of a lift distribution.

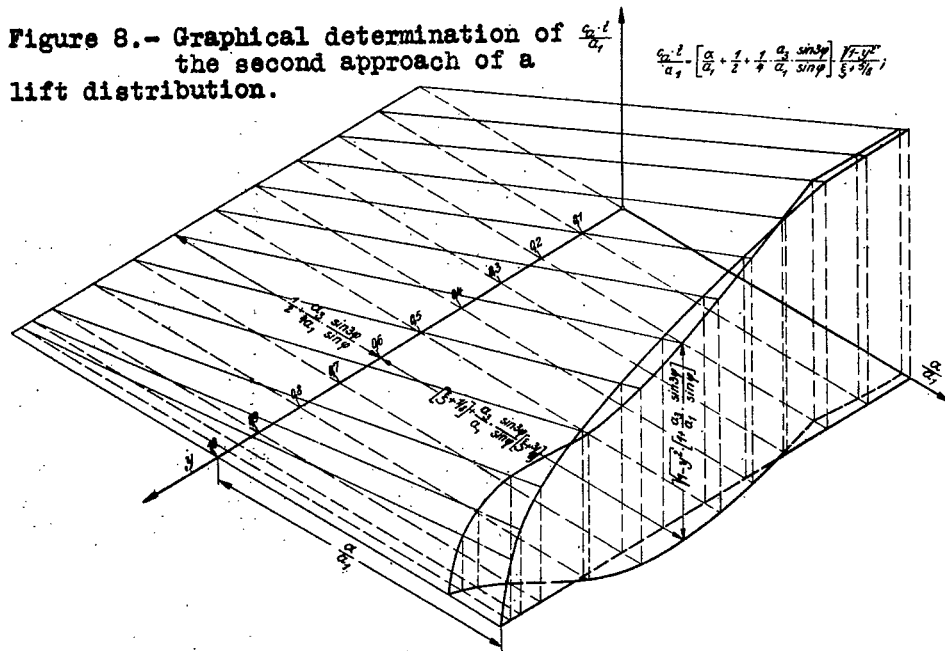


Figure 5.- Separation of angle of roll ϵ in mean angle of roll α_{Q_m} and angle of twist in roll ΔQ .

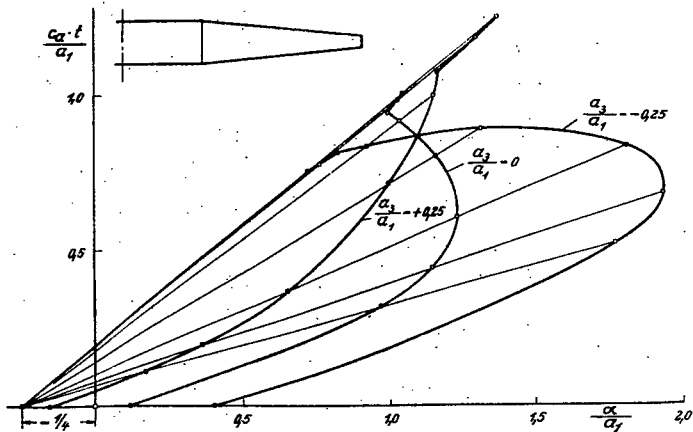
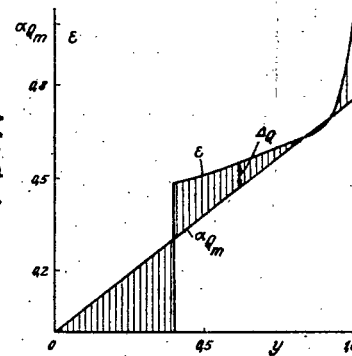


Figure 6.- Connection between lift distribution and angle of attack for two-term lift distribution.

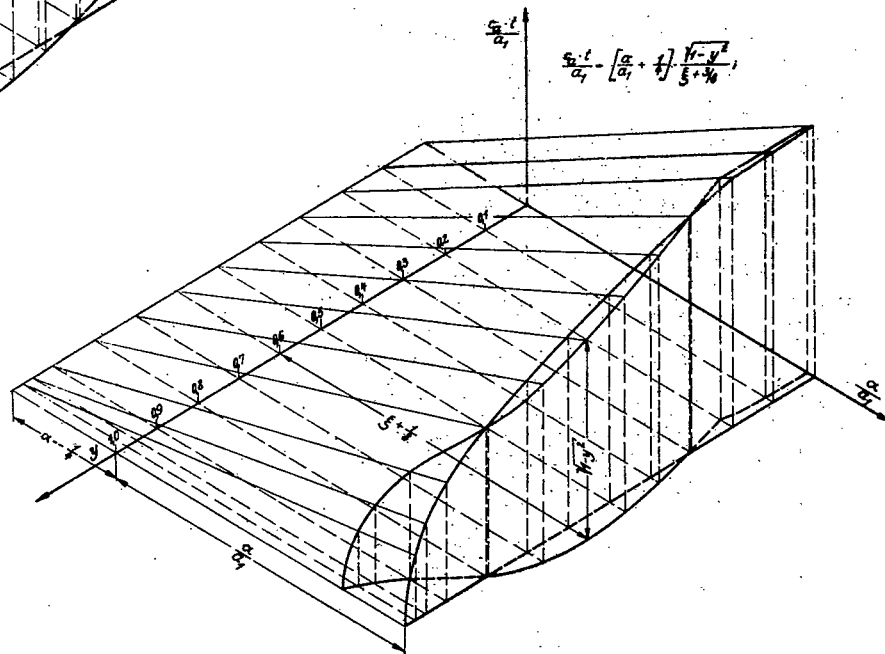


Figure 7.- Graphical determination of the first approach of a lift distribution.

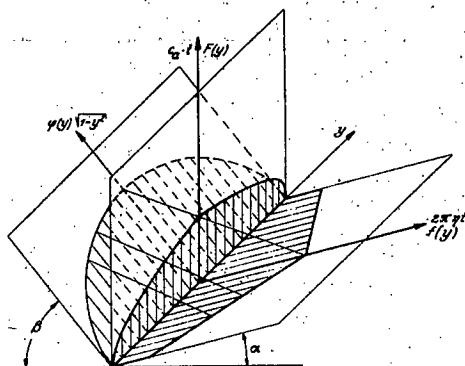


Figure 10.- Graphical determination
of a normal roll dis-
tribution.

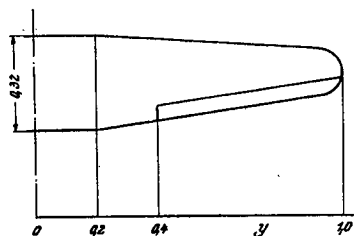


Figure 11.- Wing contour
used in the
example.

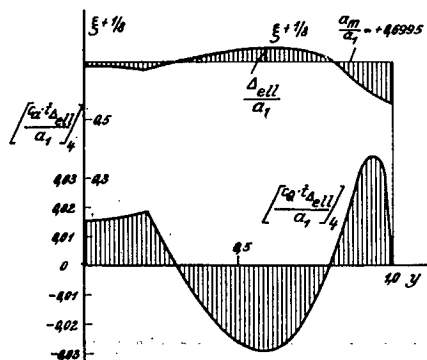
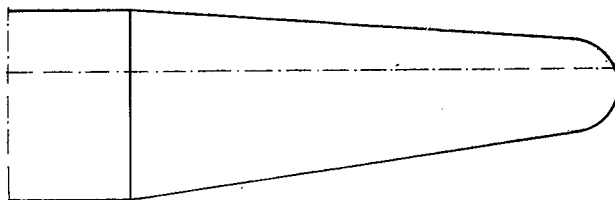


Figure 12.- Elliptical zero distribution and corresponding twist as used in the example.

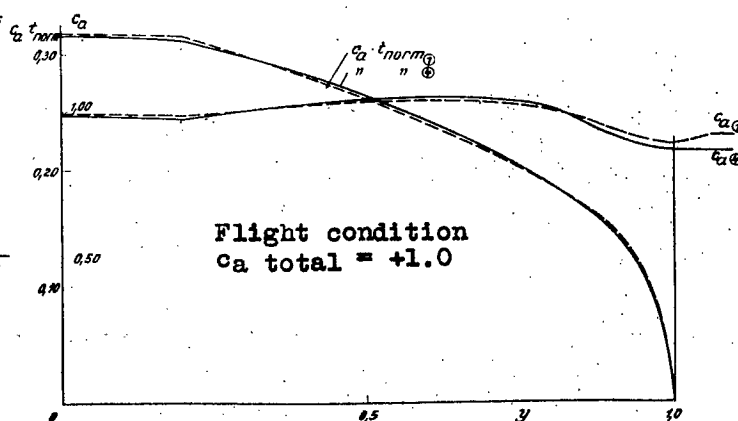


Figure 13.- Results of lift distribution in first and fourth approach.

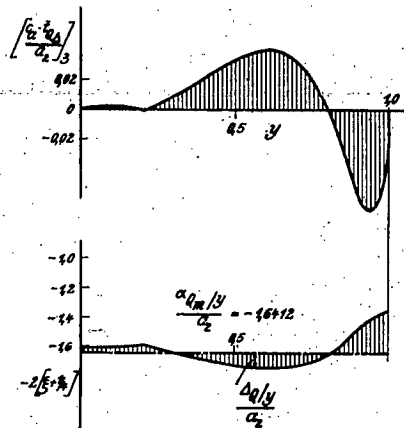


Figure 14.- Elliptical zero distribution in roll and separation into mean angle of roll and elliptical angle of twist in roll.

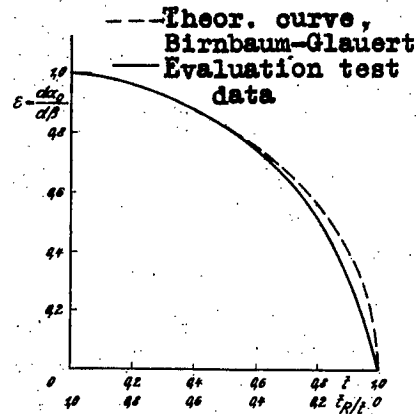


Figure 15.- Theoretical and experimental elevator effect within range of small control deflections.

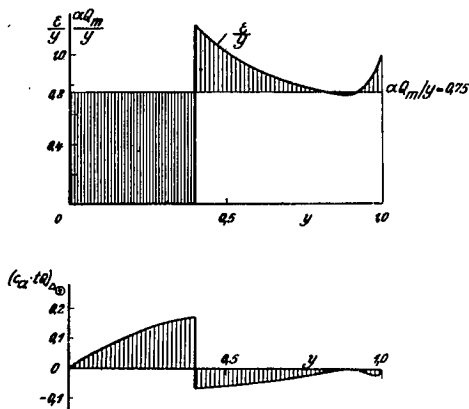


Figure 16.- Angle of twist in roll and zero distribution in roll of aileron.

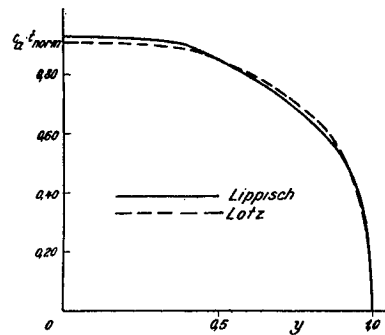


Figure 18.- Comparison of data on normal distributions of the Flavag III wing.

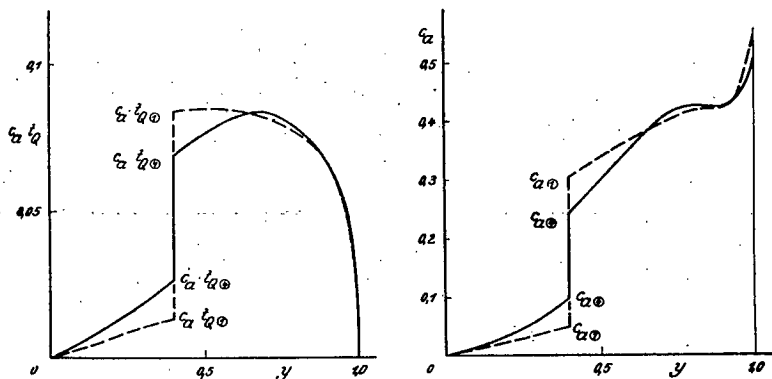


Figure 17.- First and fourth approach of aileron calculation.

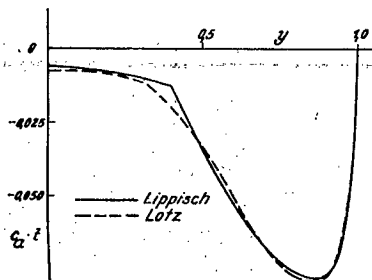


Figure 19.- Comparison of data on twisting effect for the Flavag III wing.

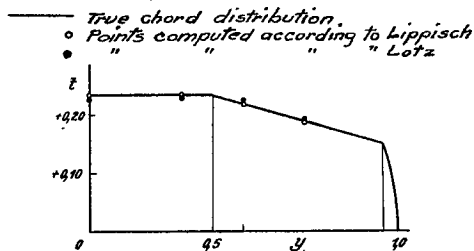


Figure 21.- Comparison of true chord distribution based on results of lift distribution analysis.

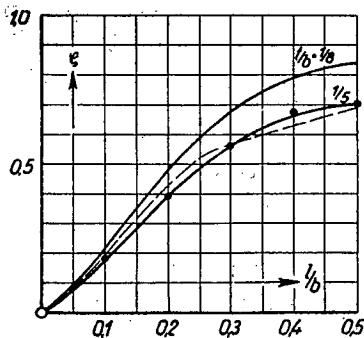


Figure 23.- Rolling moment coefficient of rectangular wing with aileron according to Lotz (ZFM 1931). The small circles are those of Lippisch obtained as first approach.

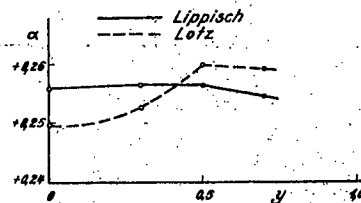


Figure 20.- Comparison of both check calculations.

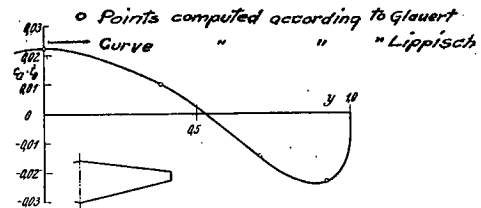


Figure 22.- Comparative calculation of a zero distribution.

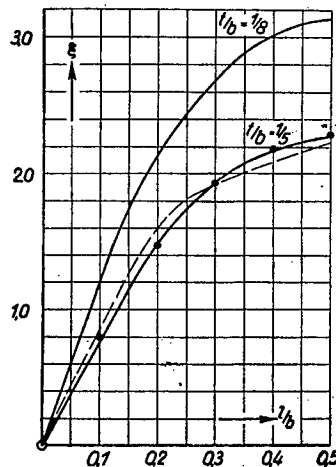


Figure 24.- Induced yawing moment coefficient of rectangular wing with aileron (Lotz ZFM 1931). The small circles are those of Lippisch obtained as first approach.

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